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Cantilever Boundary Condition, Deflections, and Stresses of Sandwich Beams

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An extended set of Timoshenko beam equations is presented, which results in a sixth-order system and allows the three boundary conditions of zero deflection, zero slope, and zero rotation angle to be satisfied at a cantilever end. The analysis assumes the core material of the sandwich provides shear stiffness, but negligible extensional stiffness to the beam, and the two skins remain equidistant from one another. The equations are found to depend on essentially two nondimensional parameters, a shear parameter g, and a skin-core thickness ratio h/h_c. Closed-form solutions are given for the deflections and stresses of an end-loaded, cantilever sandwich beam. Trends over a wide range of shear parameter g indicate that the sandwich behavior varies smoothly from that of a conventional Euler beam at large g to two separate Euler beams of the skins at low g. In the region of the cantilever end, an edge zone develops, which transitions the beam from the conventional Timoshenko behavior to a behavior that satisfies all three boundary conditions there. Simple estimates of the extent of the edge zone are given. The edge zone also causes a redistribution of the normal and shear stresses occurring there. Extension to vibrations of these beams is indicated.

Nomenclature

A	=	bh_c
A_1	=	constant defined by Eq. (62)
B_1, B_2, B_3	=	beam stiffnesses; Eqs. (13–15)
$\bar{B}_1, \bar{B}_2, \bar{B}_3$	=	nondimensional beam stiffnesses; Eqs. (34)
b	=	beam width
C_A, C_B	=	functions defined by Eqs. (94) and (95)
C_i	=	constants in Eq. (45)
C_0	=	constant defined by Eq. (59)
D_i	=	constants in Eq. (46)
E	=	modulus of elasticity of skin
F	=	axial force in beam
f	=	vertical load per unit length
G	=	shear modulus of core
g	=	GAL^2/EI
$\bar{g}$	=	g/π^2
h	=	skin thickness
h_c	=	core thickness
I	=	beam moment of inertia
I_c	=	mass moment of inertia of core
L	=	length of beam
M	=	bending moment in beam
m	=	mass per unit length
P	=	end load on beam
$\bar{P}$	=	PL^2/EI
S	=	shear force in beam
S_x	=	nondimensional normal stress in skin; Eq. (89)
T_c	=	nondimensional shear stress in core; Eq. (92)
T_s	=	nondimensional shear stress in skin; Eq. (93)
U	=	internal potential energy
u, w	=	horizontal, vertical displacements
W	=	w/L
x, z	=	horizontal, vertical coordinates
x_E	=	extent of edge zone
z_j	=	location of skin-core junction
z_u	=	location of upper skin surface
α	=	rotation angle
β	=	$1.73205\sqrt{g/\bar{B}_2}$
γ_{xz}	=	shear strain
ε_x	=	normal strain
μ_i	=	constants defined by Eq. (A14)
ξ	=	x/L
ξ_E	=	extent of edge zone
ρ	=	density
σ	=	constant defined by Eq. (48)
σ_x	=	normal stress
τ_{xz}	=	shear stress
Ψ_1, Ψ_2	=	functions defined by Eqs. (60) and (61)

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I. Introduction

FOR sandwich beams the shearing action of the softer core material is often accounted for by considering the overall sandwich beam as a Timoshenko beam, where normals to the midline need not remain normal to the midline after deformation.[1–3] This is accomplished by introducing another independent variable, rotation angle α, in addition to the usual midline deflection w. When considering the boundary conditions for the cantilever end of such a sandwich beam, however, a problem develops. Because the Timoshenko beam equations are fourth order and admit only two boundary conditions at each end, one can specify zero deflection w and zero rotation angle α at the boundary. However, the remaining physical condition of zero slope $\mathrm{d}w/\mathrm{d}x$ is not satisfied and results in a small slope there. Alternatively, one can specify zero deflection and zero slope, which would then result in a small rotation angle there.

The present paper examines the significance of this cantilever boundary condition and provides a simple extended set of Timoshenko beam equations, which result in a sixth-order system, allowing three boundary conditions at each end. Solutions are given for an end-loaded sandwich beam for core shears varying from zero to infinity and compared with the conventional Timoshenko beam formulation.

Further background on the general behavior of sandwich beams can be found in the classic books by Plantema[4] and Allen.[5] More recent articles by Frostig and Baruch[6] and Frostig et al.[7] consider additional effects of transverse compression of the soft core and normal peeling stresses between core and skins. Perel and Palazotto[8] present a detailed study of effects of through-the-thickness strains and stresses in these beams. Recent surveys of the basic analyses, methods, and literature of sandwich construction are given in the books by Vinson[9] and Zenkert.[10]

II. Sandwich Beam Formulation

The sandwich beam here is composed of a top and bottom skin, each of thickness h, joined together by a core material of thickness h_c, as shown in Fig. 1. The skins act as typical Euler beams with an elastic modulus E and always remain equidistant from one another (antiplane core). The core material is assumed to have only a shear modulus G, its elastic modulus E being considered negligible compared to the E of the skins.

From Fig. 1 one can express the horizontal displacement u of any point of the top skin, core, and bottom skin as

$$u_T = (h_c/2)\alpha - (z - h_c/2)w' \tag{1}$$

$$u_c = z\alpha \tag{2}$$

$$u_B = -(h_c/2)\alpha - (z + h_c/2)w' \tag{3}$$

where z is the distance from the midline and $()'$ represents the derivative $\mathrm{d}/\mathrm{d}x$. The corresponding normal and shear strains are

$$\varepsilon_x = \frac{\partial u}{\partial x} \tag{4}$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \tag{5}$$

Placing the displacements u into the preceding and introducing the beam stress-strain relations gives the normal stresses in the top and bottom skins and the shear stress in the core as

$$(\sigma_x)_T = E\varepsilon_x = E[(h_c/2)(\alpha' + w'') - zw''] \tag{6}$$

$$(\tau_{xz})_c = G\gamma_{xz} = G(\alpha + w') \tag{7}$$

$$(\sigma_x)_B = E\varepsilon_x = E[-(h_c/2)(\alpha' + w'') - zw''] \tag{8}$$

The normal stress in the core is assumed zero, while the shear stresses in the top and bottom skins are found, as in Euler beam theory, from σ_x through the stress equilibrium equation

$$\frac{\partial \tau_{xz}}{\partial z} = -\frac{\partial \sigma_x}{\partial x} \tag{9}$$

Upon introducing the normal stress $(\sigma_x)_T$ from Eq. (6) into the preceding and integrating from the upper surface $z = h_c/2 + h$ down to an arbitrary point z, one obtains the shear stress in the top skin as

$$(\tau_{xz})_T = E\Big\{(h_c/2)[h_c/2 + h - z](\alpha'' + w''') - \left(\tfrac{1}{2}\right)\left[(h_c/2 + h)^2 - z^2\right]w'''\Big\} \tag{10}$$

The shear stress in the bottom skin is given by $[\tau_{xz}(z)]_B = [\tau_{xz}(-z)]_T$.

The bending moment M and axial force F can be found by integrating the normal stresses in Eqs. (6–8) over the cross section from $z = -(h_c/2 + h)$ to $(h_c/2 + h)$ to give

$$M = -\int z\sigma_x b\,\mathrm{d}z = -(B_1 + B_2)\alpha' + (B_2 + B_3)w'' \tag{11}$$

$$F = \int \sigma_x b\,\mathrm{d}z = 0 \tag{12}$$

where the stiffness constants B_i are defined as

$$B_1 = \left(\tfrac{1}{2}\right)Ebh_c^2h \tag{13}$$

$$B_2 = \left(\tfrac{1}{2}\right)Ebh_ch^2 \tag{14}$$

$$B_3 = \left(\tfrac{2}{3}\right)Ebh^3 \tag{15}$$

The total EI of the sandwich beam can be expressed as

$$EI = Eb\left[(h_c + 2h)^3 - h_c^3\right]/12 = B_1 + 2B_2 + B_3 \tag{16}$$

Although the total axial force F on the cross section is zero, it is important for subsequent boundary condition considerations to distinguish the individual contributions from the top and bottom skins. For the top skin alone

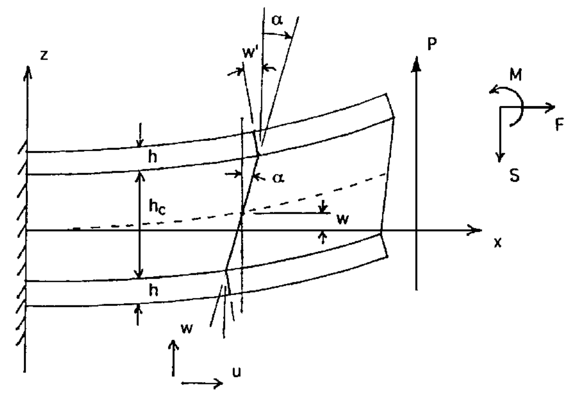

Fig. 1 Sandwich beam layout.

$$F_T = (1/h_c)(B_1\alpha' - B_2w'') \tag{17}$$

whereas for the bottom skin one has $F_B = -F_T$.

The shear force S can be found by integrating the shear stresses over the top skin, core, and bottom skin of the cross section to give, after much algebra,

$$S = -\int \tau_{xz}b\,\mathrm{d}z = -B_2\alpha'' + B_3w''' - GA(\alpha + w') \tag{18}$$

where the B_i are defined in Eqs. (13–15) and $A = bh_c$ represents the area of the core.

The overall equilibrium equations for small deflections of this sandwich beam can be obtained by looking at a small section $\mathrm{d}x$ of the beam shown in Fig. 1. From equilibrium of forces and of moments, one obtains the usual equations

$$S' = f \tag{19}$$

$$M' = S \tag{20}$$

Placing the overall M and S from Eqs. (11) and (18) into the preceding results in the two basic equations

$$B_3w^{\mathrm{IV}} - B_2\alpha''' - GA(w'' + \alpha') = f \tag{21}$$

$$B_2w''' - B_1\alpha'' + GA(w' + \alpha) = 0 \tag{22}$$

These are the extended Timoshenko beam equations for a sandwich beam. They represent a sixth-order system of equations, which now admit three boundary conditions at each end and can accommodate the cantilever boundary condition. If one drops the B_2 and B_3 terms in these equations and replaces the B_1 by EI, they will reduce to the conventional Timoshenko beam equations. (In essence, this lumps the separate B_2 and B_3 terms in with the B_1 term.)

The limiting cases of these extended Timoshenko beam equations are of interest. If one places the second equation into the first, one obtains

$$(B_2 + B_3)w^{\mathrm{IV}} - (B_1 + B_2)\alpha''' = f \tag{23}$$

For large values of GA, the third term of Eq. (22) dominates, and hence one has the relation $\alpha = -w'$. Placing this into Eq. (23) and using the stiffness identity Eq. (16), one gets the simple beam bending equation, $EI\,w^{\mathrm{IV}} = f$. For small values of GA, the third term of Eq. (22) is negligible, and one has the relation $\alpha'' = (B_2/B_1)w'''$. Placing this into Eq. (23) and using the definitions of the B_i given in Eqs. (13–15), one again gets the simple beam bending equation $(\frac{1}{6})Ebh^3w^{\mathrm{IV}} = f$, where now the bending stiffness represents the sum of the individual stiffnesses of the two skins of the sandwich beam.

III. Boundary Conditions

At the edge of a sandwich beam, one can apply a shear force S_A and a moment M_A. The moment is assumed to be applied by a horizontal force F_A acting at a distance z_A above the midline together with its oppositely directed companion on the bottom skin, to give an applied moment:

$$M_A = -2z_A F_A \tag{24}$$

Because of the general trapezoidal (rather than the usual triangular) nature of the stress distribution σ_x, one must specify the force F as well as the moment M in order to match the edge stress distribution σ_x. Making use of Eqs. (17), (11), (18), and (24) results in the three force boundary conditions at an edge:

$$B_1\alpha' - B_2 w'' = h_c F_A = -(h_c/2z_A)M_A \tag{25}$$

$$-(B_1 + B_2)\alpha' + (B_2 + B_3)w'' = M_A \tag{26}$$

$$-B_2\alpha'' + B_3 w''' - GA(\alpha + w') = S_A \tag{27}$$

The preceding shear condition can be rearranged by introducing Eq. (22) to give the simpler form:

$$-(B_1 + B_2)\alpha'' + (B_2 + B_3)w''' = S_A \tag{28}$$

Also, if the applied moment $M_A = 0$, then the first two boundary conditions, Eqs. (25) and (26), reduce to

$$\alpha' = 0 \tag{29}$$

$$w'' = 0 \tag{30}$$

because these equations become homogeneous and their determinant is not equal to zero.

The three geometric boundary conditions at an edge for these sandwich beams are

$$w = w_A \tag{31}$$

$$\alpha = \alpha_A \tag{32}$$

$$w' = w'_A \tag{33}$$

For completeness, an energy formulation of the basic differential equations and boundary conditions for this sandwich beam is presented in the Appendix. Also, the inertia terms required for vibrations of these extended Timoshenko beams is presented there.

IV. Solution for an End-Loaded Cantilever Beam

The preceding theory is applied to a uniform cantilever sandwich beam, which is loaded by a vertical force P, as shown in Fig. 1. The beam is of width b and of length L and has a core thickness h_c and a skin thickness h. The governing differential equations are given by Eqs. (21) and (22) with $f = 0$, and the boundary conditions are given at $x = 0$ by Eqs. (31–33), and at $x = L$ by Eqs. (28–30) with $S_A = -P$. Before solving, it is convenient to nondimensionalize these equations as follows:

$$W = \frac{w}{L}, \qquad \xi = \frac{x}{L}, \qquad \frac{\mathrm{d}}{\mathrm{d}\xi} = L\frac{\mathrm{d}}{\mathrm{d}x}, \qquad g = \frac{GAL^2}{EI}$$

$$\bar{P} = \frac{PL^2}{EI}, \qquad \bar{B}_1 = \frac{B_1}{EI} = \frac{\frac{1}{2}}{\left(\frac{1}{2}\right) + (h/h_c) + \left(\frac{2}{3}\right)(h/h_c)^2}$$

$$\bar{B}_2 = \frac{B_2}{EI} = \frac{\left(\frac{1}{2}\right)(h/h_c)}{\left(\frac{1}{2}\right) + (h/h_c) + \left(\frac{2}{3}\right)(h/h_c)^2}$$

$$\bar{B}_3 = \frac{B_3}{EI} = \frac{\left(\frac{2}{3}\right)(h/h_c)^2}{\left(\frac{1}{2}\right) + (h/h_c) + \left(\frac{2}{3}\right)(h/h_c)^2}$$

$$\bar{B}_1 + 2\bar{B}_2 + \bar{B}_3 = 1, \qquad \bar{B}_1\bar{B}_3 - (\bar{B}_2)^2 = \left(\tfrac{1}{3}\right)(\bar{B}_2)^2 \tag{34}$$

Note: In Secs. IV and V the notation ( )′ will now be used to designate the nondimensional derivative $\mathrm{d}/\mathrm{d}\xi$, rather than the dimensional $\mathrm{d}/\mathrm{d}x$ that was used in Secs. II and III. The nondimensional form of the governing equations and the boundary conditions then become

$$\bar{B}_3 W^{\mathrm{IV}} - \bar{B}_2\alpha''' - g(W'' + \alpha') = 0 \tag{35}$$

$$\bar{B}_2 W''' - \bar{B}_1\alpha'' + g(W' + \alpha) = 0 \tag{36}$$

at $\xi = 0$

$$W = 0 \tag{37}$$

$$W' = 0 \tag{38}$$

$$\alpha = 0 \tag{39}$$

at $\xi = 1$

$$W'' = 0 \tag{40}$$

$$\alpha' = 0 \tag{41}$$

$$(\bar{B}_1 + \bar{B}_2)\alpha'' - (\bar{B}_2 + \bar{B}_3)W''' = \bar{P} \tag{42}$$

The sandwich beam equations here are seen to be characterized by two nondimensional parameters, namely, the skin-core thickness ratio h/h_c and the shear parameter g.

To solve these equations, one assumes exponential solutions for W and α in the form $e^{\lambda\xi}$ to arrive at the sixth-order characteristic equation

$$(\bar{B}_2)^2\lambda^6 - 3g\lambda^4 = 0 \tag{43}$$

which yields six roots $\lambda = +\beta, -\beta, 0, 0, 0, 0$, where

$$\beta = 1.73205\sqrt{g}\big/\bar{B}_2 \tag{44}$$

The corresponding solutions for W and α are

$$W = C_1 e^{\beta\xi} + C_2 e^{-\beta\xi} + C_3 + C_4\xi + C_5\xi^2 + C_6\xi^3 \tag{45}$$

$$\alpha = D_1 e^{\beta\xi} + D_2 e^{-\beta\xi} + D_3 + D_4\xi + D_5\xi^2 + D_6\xi^3 \tag{46}$$

The constants D_i are related to the C_i by matching appropriate coefficients in Eq. (36) to give

$$D_1 = \beta\sigma C_1, \qquad D_2 = -\beta\sigma C_2$$

$$D_3 = -C_4 - 6C_6(\bar{B}_1 + \bar{B}_2)/g, \qquad D_4 = -2C_5$$

$$D_5 = -3C_6, \qquad D_6 = 0 \tag{47}$$

where in the preceding

$$\sigma = \frac{\bar{B}_2\beta^2 + g}{\bar{B}_1\beta^2 - g} = \frac{\bar{B}_2 + \bar{B}_2^2\big/3}{\bar{B}_1 - \bar{B}_2^2\big/3} \tag{48}$$

and σ is seen to depend only on h/h_c and not on β or g. Placing W and α together with Eqs. (47) into the boundary conditions results in six equations for the six consants C_i, namely,

$$C_1 + C_2 + C_3 = 0 \tag{49}$$

$$\beta C_1 - \beta C_2 + C_4 = 0 \tag{50}$$

$$\beta\sigma C_1 - \beta\sigma C_2 - C_4 - (6/g)(\bar{B}_1 + \bar{B}_2)C_6 = 0 \tag{51}$$

$$\beta^2 e^{\beta} C_1 + \beta^2 e^{-\beta} C_2 + 2C_5 + 6C_6 = 0 \tag{52}$$

$$\beta^2\sigma e^{\beta} C_1 + \beta^2\sigma e^{-\beta} C_2 - 2C_5 - 6C_6 = 0 \tag{53}$$

$$-6C_6 = \bar{P} \tag{54}$$

where use was made of the identity

$$\sigma(\bar{B}_1 + \bar{B}_2) - (\bar{B}_2 + \bar{B}_3) = 0 \tag{55}$$

The relation between C_1 and C_2 is readily obtained from Eqs. (52) and (53). Then solving for the C_i from these equations gives after much algebra

$$W = \bar{P}\left\{-(A_1/\beta)[C_0 - \Psi_1(\xi)] + A_1\xi + \left(\tfrac{1}{2}\right)\xi^2 - \left(\tfrac{1}{6}\right)\xi^3\right\} \tag{56}$$

$$\alpha = \bar{P}\left\{A_1\sigma[1 - \Psi_2(\xi)] - \xi + \left(\tfrac{1}{2}\right)\xi^2\right\} \tag{57}$$

$$W' = P\left\{A_1[1 - \Psi_2(\xi)] + \xi - \left(\tfrac{1}{2}\right)\xi^2\right\} \tag{58}$$

where

$$C_0 = \frac{1 - e^{-2\beta}}{1 + e^{-2\beta}} \tag{59}$$

$$\Psi_1(\xi) = \frac{e^{-\beta\xi} - e^{-\beta(2-\xi)}}{1 + e^{-2\beta}} \tag{60}$$

$$\Psi_2(\xi) = \frac{e^{-\beta\xi} + e^{-\beta(2-\xi)}}{1 + e^{-2\beta}} \tag{61}$$

$$A_1 = \frac{\bar{B}_1 + \bar{B}_2}{g(1 + \sigma)} \tag{62}$$

and σ and β are defined by Eqs. (48) and (44), respectively, whereas g and $\bar{B}_2$ are defined by the nondimensional parameters [Eqs. (34)].

Equations (56–58) represent the general deflection, rotation angle and slope of the sandwich beam for arbitraty values of g and h/h_c. For large values of the parameter β [and hence, g from Eq. (44)] such that $\beta > 5$, the double exponentials $\Psi_i(\xi)$ can be reduced to give the simpler expressions

$$W = \bar{P}\left[-(A_1/\beta)(1 - e^{-\beta\xi}) + A_1\xi + \left(\tfrac{1}{2}\right)\xi^2 - \left(\tfrac{1}{6}\right)\xi^3\right] \tag{63}$$

$$\alpha = \bar{P}\left[A_1\sigma(1 - e^{-\beta\xi}) - \xi + \left(\tfrac{1}{2}\right)\xi^2\right] \tag{64}$$

$$W' = \bar{P}\left[A_1(1 - e^{-\beta\xi}) + \xi - \left(\tfrac{1}{2}\right)\xi^2\right] \tag{65}$$

This form clearly indicates an exponentially decaying edge zone near $\xi = 0$, which satisfies all three proper boundary conditions at $\xi = 0$. This edge zone dies out to 5% of its value when $\beta\xi = 3$ so that the extent of the edge zone ξ_E can be estimated simply as

$$\xi_E = x_E/L = 3/\beta \tag{66}$$

Alternatively, using the definitions of β, g, and $\bar{B}_2$, one can also express this edge zone region in terms of skin thickness as

$$x_E/h = 1.225\sqrt{\bar{B}_2}\sqrt{E/G} \tag{66a}$$

It is interesting to compare the sandwich solutions given by Eqs. (63–65) with those of the conventional Timoshenko beam solution. As already mentioned, the conventional Timoshenko equations would be given by Eqs. (35) and (36) with $\bar{B}_2 = \bar{B}_3 = 0$ and $\bar{B}_1 = 1$. The resulting equations are usually solved statically by transforming to two new variables W_b and W_s such that

$$W = W_b + W_s \tag{67}$$

$$\alpha = -W_b' \tag{68}$$

This reduces Eqs. (35) and (36) to two uncoupled beam equations, namely, the bending equation $W_b^{\mathrm{IV}} = 0$, and the shear equation $gW_s'' = 0$, which can then be solved separately. The boundary conditions are similarly reduced to $W_b = 0$, $W_b' = 0$, $W_s = 0$ at $\xi = 0$, and $W_b'' = 0$, $W_b''' = -\bar{P}$, $W_s = \bar{P}/g$ at $\xi = 1$. (Note here that the boundary condition $W_s' = 0$ is not satisfied at $\xi = 0$.) The resulting conventional Timoshenko solutions for this end-loaded cantilever are readily found to be

$$W = \bar{P}\left[(1/g)\xi + \left(\tfrac{1}{2}\right)\xi^2 - \left(\tfrac{1}{6}\right)\xi^3\right] \tag{69}$$

$$\alpha = \bar{P}\left[-\xi + \left(\tfrac{1}{2}\right)\xi^2\right] \tag{70}$$

$$W' = \bar{P}\left[(1/g) + \xi - \left(\tfrac{1}{2}\right)\xi^2\right] \tag{71}$$

As already mentioned, these do not satisfy the boundary condition $W' = 0$ at $\xi = 0$. However, the extended Timoshenko solutions given by Eqs. (63–65) for large β are seen to reduce to these equations and additionally have the exponentially decaying edge zones to satisfy all three boundary conditions at $\xi = 0$. The extent of this edge zone can be estimated from Eqs. (66) or (66a). For low values of β (and hence, shear parameter g), the more accurate extended Timoshenko solutions [Eqs. (56–58)] should be used.

The normal stress in the top skin $(\sigma_x)_T$ is given by Eq. (6) as

$$(\sigma_x)_T = (E/L)[(h_c/2)(\alpha' + W'') - zW''] \tag{72}$$

where the notation $()'$ represents $\mathrm{d}/\mathrm{d}\xi$ in Sec. IV. Placing in the expressions for α and W from Eqs. (56–58) will give, after some algebra,

$$(\sigma_x)_T = (PLz_u/I)\{-(1 - \xi)(z/z_u) + A_1\beta\Psi_1(\xi)[(1 + \sigma)/(1 + 2h/h_c) - (z/z_u)]\} \tag{73}$$

where

$$A_1\beta = \frac{(\bar{B}_1 + \bar{B}_2)\beta}{g(1 + \sigma)} \tag{74}$$

and (PLz_u/I) represents the maximum stress at the upper surface of the top skin at the beam root, according to conventional beam theory. At the upper part of the top skin, $z = z_u = h + h_c/2$, and at the skin-core junction $z = z_j = h_c/2$; Eq. (73) gives, respectively,

$$\sigma_{xu} = (PLz_u/I)[-(1 - \xi) + A_1\beta\Psi_1(\xi) \times (\sigma - 2h/h_c)/(1 + 2h/h_c)] \tag{75}$$

$$\sigma_{xj} = (PLz_u/I)[-(1 - \xi)/(1 + 2h/h_c) + A_1\beta\Psi_1(\xi)\sigma/(1 + 2h/h_c)] \tag{76}$$

and the stress is distributed linearly between these two values. In the bottom skin the stresses are negative of those just shown, that is, $[\sigma_x(z)]_B = -[\sigma_x(-z)]_T$. As in the case of the deflections, for large values of the parameter β (and hence g) such that $\beta > 5$ the preceding expressions reduce more simply to

$$\sigma_{xu} = (PLz_u/I)\left[-(1 - \xi) + A_1\beta e^{-\beta\xi} \times (\sigma - 2h/h_c)/(1 + 2h/h_c)\right] \tag{77}$$

$$\sigma_{xj} = (PLz_u/I)\left[-(1 - \xi)/(1 + 2h/h_c) + A_1\beta e^{-\beta\xi}\sigma\big/(1 + 2h/h_c)\right] \tag{78}$$

The $(1 - \xi)$ term in these expressions indicates the conventional beam stress, while the exponential terms give a correction to this value.

The shear stress in the core $(\tau_{xz})_c$ is given by Eq. (7) as

$$(\tau_{xz})_c = G(\alpha + W') \tag{79}$$

where the notation $()'$ again represents $\mathrm{d}/\mathrm{d}\xi$ in Sec. IV. Placing the expressions for α and W' from Eqs. (57) and (58) into Eq. (79) will give

$$(\tau_{xz})_c = (P/A)\{(\bar{B}_1 + \bar{B}_2)[1 - \Psi_2(\xi)]\} \tag{80}$$

where P/A represents the shear stress if the shear load were distributed completely over the core area alone. For large values of β such that $\beta > 5$, the preceding reduces more simply to

$$(\tau_{xz})_c = (P/A)[(\bar{B}_1 + \bar{B}_2)(1 - e^{-\beta\xi})] \tag{81}$$

The effect of the edge zone near $\xi = 0$ is readily apparent here and shows that the core carries no shear stress at the root. Instead, it is all carried by the skins there. The skins also carry part of the shear load elsewhere in the beam.

The shear stress carried by the top skin $(\tau_{xz})_T$ is given by Eq. (10) as

$$(\tau_{xz})_T = (E/L^2)\left[z_j(z_u - z)(\alpha'' + W''') - \left(\tfrac{1}{2}\right)\left(z_u^2 - z^2\right)W'''\right] \tag{82}$$

where again ( )′ represents d/dξ in Sec. IV. Placing α and W directly into Eq. (82) will give

$$(\tau_{xz})_T = \left(Pz_u^2/I\right)\left(\left(\tfrac{1}{2}\right)\left[1-(z/z_u)^2\right] + A_1\beta^2\Psi_2(\xi)\left\{\left(\tfrac{1}{2}\right)\right.\right.$$
$$\left.\left.\times\left[1-(z/z_u)^2\right] - (1+\sigma)(1-z/z_u)/(1+2h/h_c)\right\}\right) \quad (83)$$

where the preceding quantities are defined as in Eq. (73) for $(\sigma_x)_T$. In the bottom skin the shear stresses are distributed symmetrically to those just shown, that is, $[\tau_{xz}(z)]_B = [\tau_{xz}(-z)]_T$. For values of $\beta > 5$, Eq. (83) reduces more simply to

$$(\tau_{xz})_T = \left(Pz_u^2/I\right)\left(\left(\tfrac{1}{2}\right)\left[1-(z/z_u)^2\right] + A_1\beta^2 e^{-\beta\xi}\left\{\left(\tfrac{1}{2}\right)\right.\right.$$
$$\left.\left.\times\left[1-(z/z_u)^2\right] - (1+\sigma)(1-z/z_u)/(1+2h/h_c)\right\}\right) \quad (84)$$

The first term $(\frac{1}{2})[1-(z/z_u)^2]$ in the preceding gives the conventional shear stress in the skin, whereas the exponential term gives a correction to this value, particularly inside the edge zone $\xi < \xi_E$. These skin shear stresses can be normalized in the same form as the core shear stresses by noting that

$$\left(Pz_u^2/I\right) = (P/A)\left(Az_u^2/I\right) \quad (85)$$

and expressing the (Az_u^2/I) factor using Eqs. (34) and (13) as

$$\frac{Az_u^2}{I} = \frac{\bar{B}_1(1+2h/h_c)^2}{2(h/h_c)} \quad (86)$$

The shear stress $(\tau_{xz})_T$ at the skin-core junction $z_j = h_c/2$ will match the constant shear stress in the core $(\tau_{xz})_c$ found earlier.

V. Application

The solution developed in the preceding section is applied to a specific beam in order to show some of its ramifications. The sandwich beam chosen has a skin-core thickness ratio $h/h_c = 0.125$, and the shear parameter g will be varied from zero to infinity. For this h/h_c the corresponding nondimensionsal stiffness parameters from Eqs. (34) are computed as $\bar{B}_1 = 0.786885$, $\bar{B}_2 = 0.098361$, and $\bar{B}_3 = 0.016394$, whereas from Eq. (48) one obtains $\sigma = 0.129796$. From Eqs. (66) and (44) the transitional value of $\beta = 5$ corresponds to an edge zone region $\xi_E = 0.6$ and to a shear parameter $g = 0.0806$. The value of g for $\beta = 5$ at other values of h/h_c is shown in Fig. 2. For values of g higher than this boundary, the simpler single exponential forms of Eqs. (63–65), (77), (78), (81), and (84) can be used. Figure 2 also gives the extent of the edge zone ξ_E for other h/h_c and g values.

Figure 3 shows the tip deflection $W_L/\bar{P}$ over a wide range of the shear parameter g from both the extended theory Eqs. (56–58) and the Timoshenko theory Eqs. (69–71). For large values of g, both theories approach the Euler beam deflection of $W_L/\bar{P} = 0.3333$. As g decreases, the tip deflection increases as a result of the shearing action. However, although the Timoshenko theory keeps increasing indefinitely, the extended theory approaches the limiting value of two smaller Euler beams whose combined $EI = (\frac{1}{6})\ Ebh^3 = (\frac{1}{4})\bar{B}_3 EI$. This gives a limiting deflection:

$$W_L/\bar{P} = 0.3333(4/\bar{B}_3) = 81.3 \quad (87)$$

for the extended theory here as $g \to 0$. The two theories agree to within 3% down to about $g = 20$, at which point the shear deflection at the tip is about 12% of the purely bending deflection $W_L/\bar{P} = 0.3333$.

Beam deflection shapes W/W_L from the extended theory Eqs. (56–58) are shown in Fig. 4. At large g the shape is essentially the Euler beam shape. As g decreases, the shape gets flatter, reflecting more shear effects. However for still smaller g, it reverts back to the Euler beam shape again. The maximum shear distortion effects occur here at about $g = 1$. The Timoshenko beam theory, Eqs. (69–71), would give the shape $W/W_L = \xi$ as $g \to 0$.

Beam slopes W'/W_L are shown in Fig. 5. Here again, at both large and small g the slopes are essentially those of Euler beams, with maximum deviations occuring around $g = 1$. The presence of the edge zone is clearly apparent for $g = 10$, 1, and 0.1 here. For these cases one has $\beta = 55.69$, 17.61, and 5.569, respectively, giving

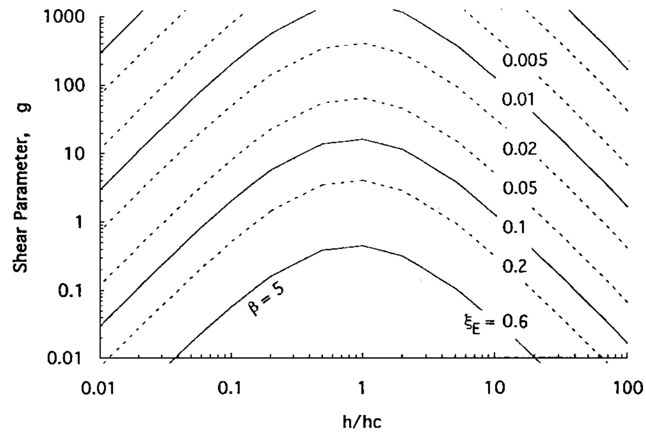

Fig. 2 Extent of edge zones.

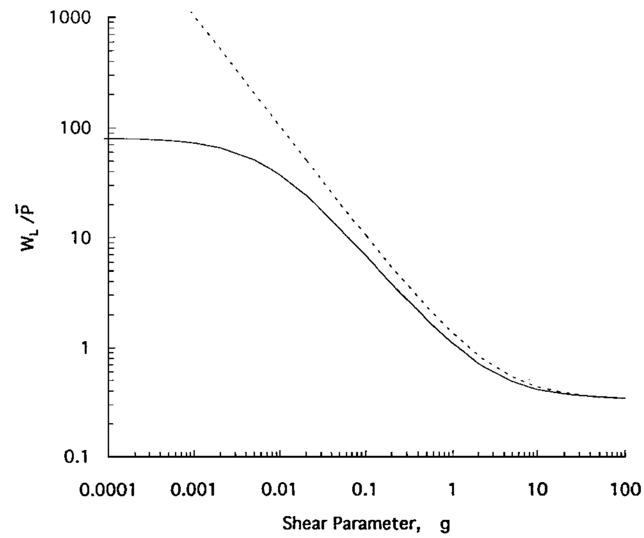

Fig. 3 Beam tip deflection $W_L/\bar{P}$ vs shear parameter g: ——, extended solution and - - - -, Timoshenko solution ($h/h_c = 0.125$).

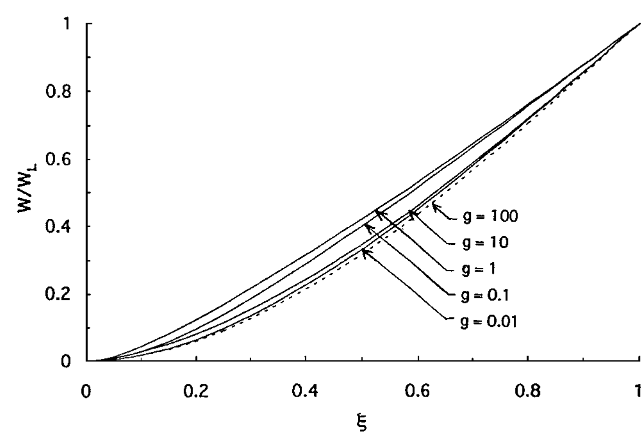

Fig. 4 Beam deflection shapes W/W_L ($h/h_c = 0.125$).

edge zone regions of $\xi_E = 0.054$, 0.170, and 0.539. The Timoshenko beam theory for these cases would result in slopes at the root $W'(0)/W_L = 0.231$, 0.750, and 0.968, respectively, and would give $W'(\xi)/W_L = 1$ as $g \to 0$.

Beam rotation angles α/W_L are shown in Fig. 6. At large g these reflect the Euler beam relation $\alpha = -W'$. As g decreases, the rotation angle decreases toward zero indicating more shearing deformation, even becoming slightly positive when $g \to 0$. This occurs because, although the centers of the top and bottom skins deflect vertically, the line joining the skin-core junctions of these skins would become slightly positive (see Fig. 1). The Timoshenko beam theory would also show this decreasing α trend with g, but would always maintain the same shape $\alpha(\xi)$ as g decreased.

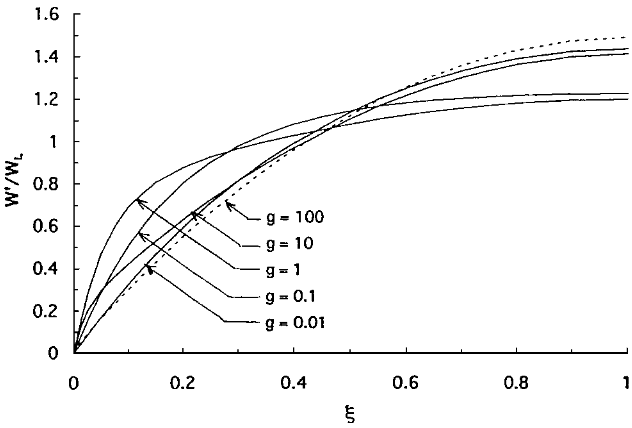

Fig. 5 Beam slopes W'/W_L ($h/h_c = 0.125$).

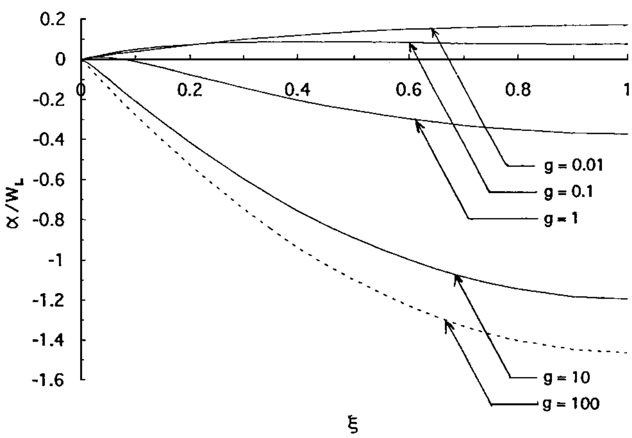

Fig. 6 Beam rotation angles α/W_L ($h/h_c = 0.125$).

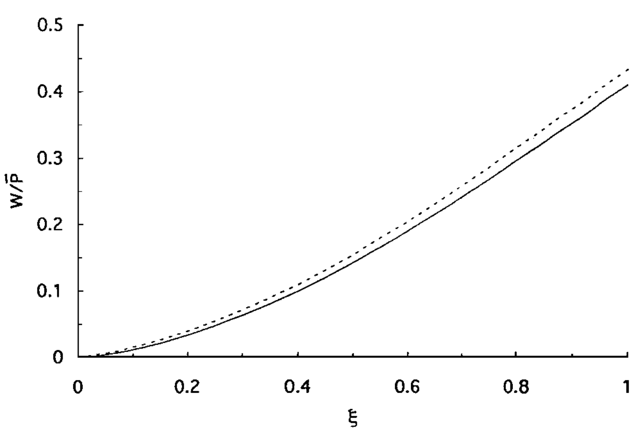

Fig. 7 Beam deflections $W/\bar{P}$ at $g = 10$: ——, extended solution and - - -, Timoshenko solution ($h/h_c = 0.125$).

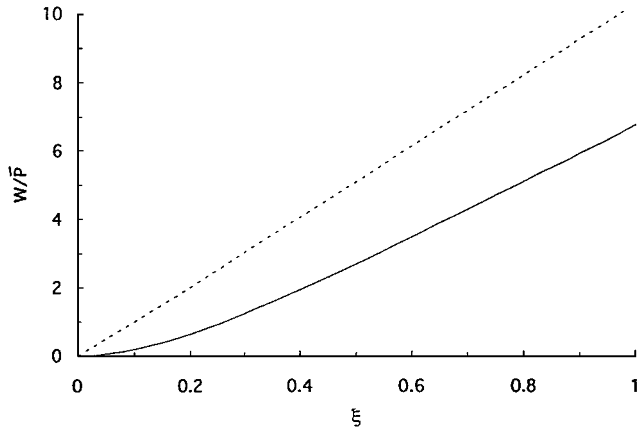

Fig. 8 Beam deflections $W/\bar{P}$ at $g = 0.1$: ——, extended solution and - - -, Timoshenko solution ($h/h_c = 0.125$).

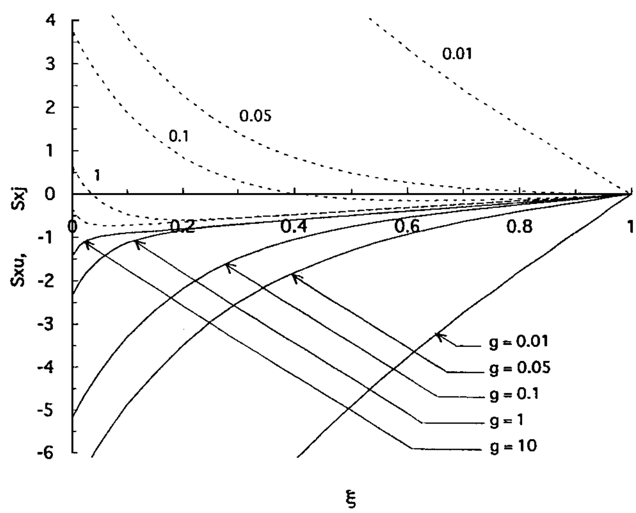

Fig. 9 Normal stresses in top skin: ——, upper surface S_{xu} and - - -, skin-core junction S_{xj} ($h/h_c = 0.125$).

Figures 7 and 8 compare the beam deflections from the extended theory and the Timoshenko theory for $g = 10$ and 0.1. At $g = 10$ the two deflections are quite similar, but at $g = 0.1$ the deflections from the Timoshenko theory are much larger and dominated by the shearing contribution of the core, whereas the extended theory allows the skins to take up much of the shearing action.

The normal stresses $(\sigma_x)_T$ in the top skin from the extended theory are given by Eqs. (73), (75), and (76). These can be written more conveniently as

$$(\sigma_x)_T = (PLz_u/I)S_x \tag{88}$$

where S_x is the nondimensional factor following the (PLz_u/I) term in these equations. Figure 9 shows the distribution of the normal stresses in the top skin at the upper surface $z_u = h + h_c/2$, and at the skin-core junction $z_j = h_c/2$ for several values of the shear parameter g. The normal stress varies linearly in the top skin between these two values. For large values of g, the extended theory stress distribution is similar to the Timoshenko beam model except near the root, where an edge zone is apparent and the stresses increase as a result of the extra bending there. The Timoshenko beam would have given stresses $S_{xu} = -1.00$ and $S_{xj} = -0.80$ at the root. As g decreases, the edge zone increases, and eventually S_{xj} reverses sign and becomes positive. As $g \to 0$, the skins begin acting as individual Euler beams of thickness h in bending. This would result in the root stresses

$$\sigma_x = \frac{(PL/2)(h/2)}{bh^3/12} = \frac{3PL}{bh^2} \tag{89}$$

which, when put into the framework of Eq. (88) with the help of the relationship $I = bh_c h^2/2\bar{B}_2$, would give

$$S_{xu} = -S_{xj} = -3/\bar{B}_2(1 + 2h/h_c) = -24.40 \tag{90}$$

Figure 10 shows the variation of these normal stresses S_{xu} and S_{xj} at the root over a wide range of the shear parameter g. Typical distributions of normal stress S_x through the skin thickness at $\xi = 0.2$ are shown in Fig. 11. The distribution at the larger value $g = 10$ is seen to be linear and would pass through $z/z_u = 0$, similar to the Timoshenko stress distribution. At the lower value $g = 0.1$, the stress distribution, although linear, changes sign between the upper surface z_u and the skin-core junction z_j.

The shear stress in the core $(\sigma_{xz})_c$ is given by Eq. (80) and can be written more conveniently as

$$(\tau_{xz})_c = (P/A)T_c \tag{91}$$

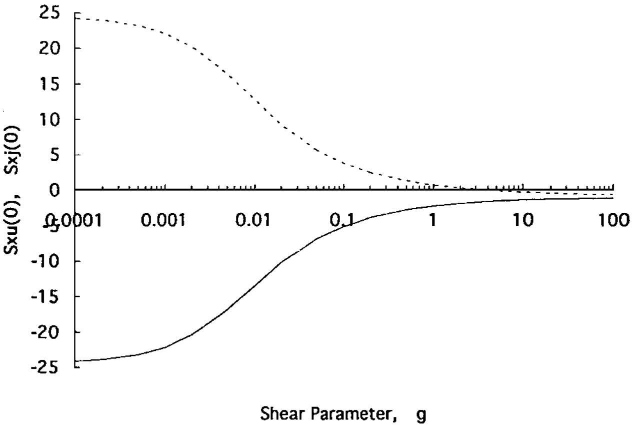

Fig. 10 Normal stresses in top skin at root vs shear parameter g: ——, upper surface S_{xu} (0) and - - - -, skin-core junction S_{xj} (0) ($h/h_c = 0.125$).

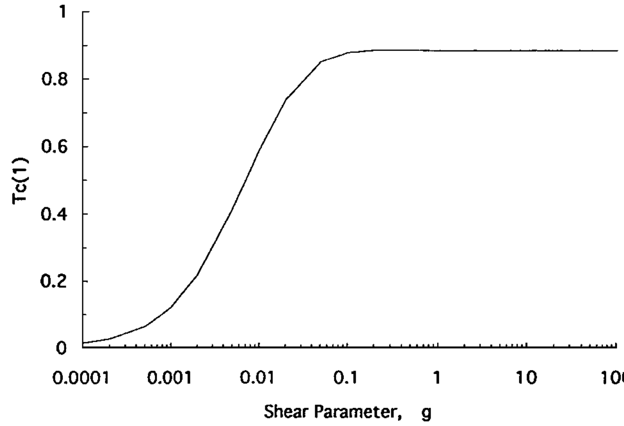

Fig. 13 Shear stresses in core at tip $T_c(1)$ vs shear parameter g ($h/h_c = 0.125$).

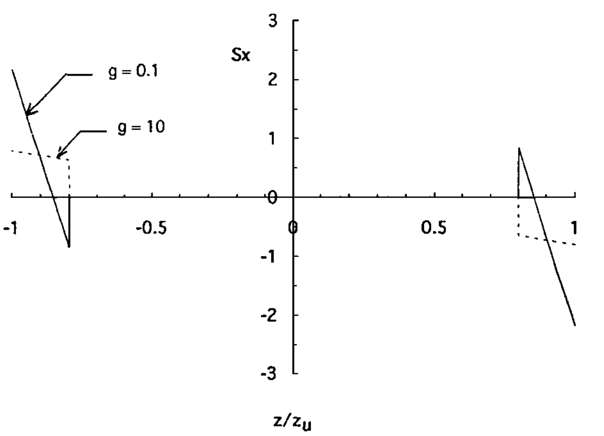

Fig. 11 Normal stress distributions in beam S_x at $x/L = 0.2$ ($h/h_c = 0.125$).

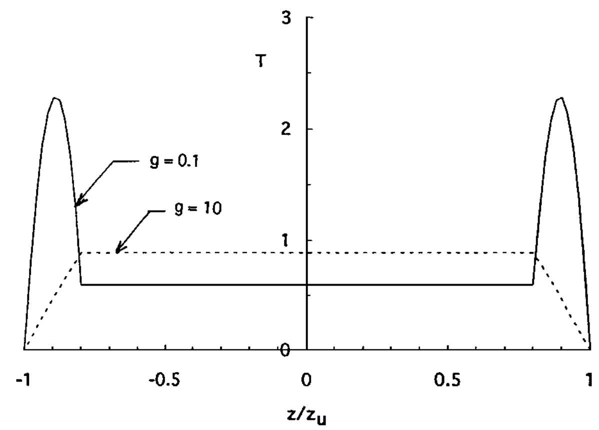

Fig. 14 Shear-stress distributions in beam T_s, T_c at $x/L = 0.2$ ($h/h_c = 0.125$).

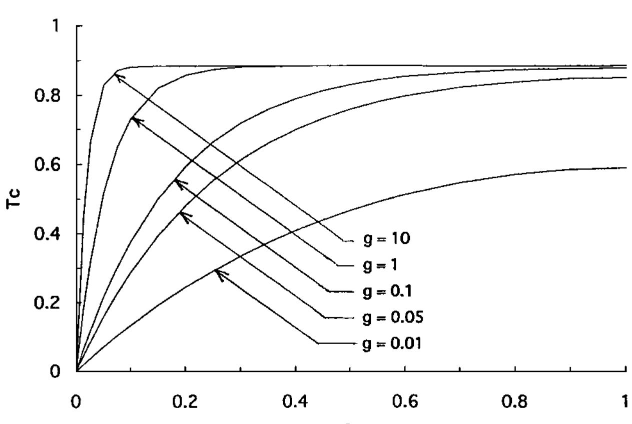

Fig. 12 Shear stresses in core T_c ($h/h_c = 0.125$).

where T_c is the nondimensional factor following the (P/A) term in that equation. Figure 12 shows the distribution of the core shear stress for several values of the shear parameter g. For large g the shear stress is constant over the length of the beam, except near the root where it drops to zero. If all of the shear load were carried by the core alone, one would have $T_c = 1$ everywhere in the beam. As g decreases, the edge zone increases, and the shear stress T_c decreases throughout the beam. As $g \to 0$, the shear load is carried be the skins alone. Figure 13 shows the variation of the core shear stress T_c at the beam tip over a wide range of the shear parameter g.

The shear stress carried by the top skin $(\tau_{xz})_T$ from the extended theory is given by Eqs. (83), (85), and (86). This can be written more conveniently as

$$(\tau_{xz})_T = (P/A)\left\{C_A\left[1 - (z/z_u)^2\right] - C_B(1 - z/z_u)\right\} = (P/A)T_s \tag{92}$$

where

$$C_A = \left[1 + A_1\beta^2\Psi_2(\xi)\right]\left(Az_u^2/I\right)/2 \tag{93}$$

$$C_B = A_1\beta^2\Psi_2(\xi)(1+\sigma)\left(Az_u^2/I\right)/(1 + 2h/h_c) \tag{94}$$

Typical distributions of the skin shear stress T_s through the skin thickness at $\xi = 0.2$ are shown in Fig. 14 . The distributions are parabolic. At the larger value $g = 10$ the maximum occurs at the skin-core junction, because the shear load is taken up mostly by the core. At the lower value $g = 0.1$, the core takes up much less load, and the skin shear stress peaks up well above the core stress to carry the shear load. The maximum shear stress in the top skin can he found by setting $d\tau_{xz}/dz = 0$ in Eq. (92) to give the location of the maximum stress as

$$(z/z_u)_m = C_B/2C_A \tag{95}$$

Placing this into Eq. (92) gives the maximum shear stress in the skin $(\tau_{xz})_m$, providing that z_m occurs above the skin-core junction. If $z_m < z_j$, the maximum skin stress occurs at the skin-core junction and is equal to the core shear stress T_c. The T_c can be obtained as before from Eq. (91), or equivalently by placing $z_j/z_u = 1/(1 + 2h/h_c)$ into Eq. (92). Figure 15 shows the maximum shear stress $(T_s)_m$ developed in the skin for any given core shear stress T_c. The smaller the T_c, the larger the $(T_s)_m$ because the

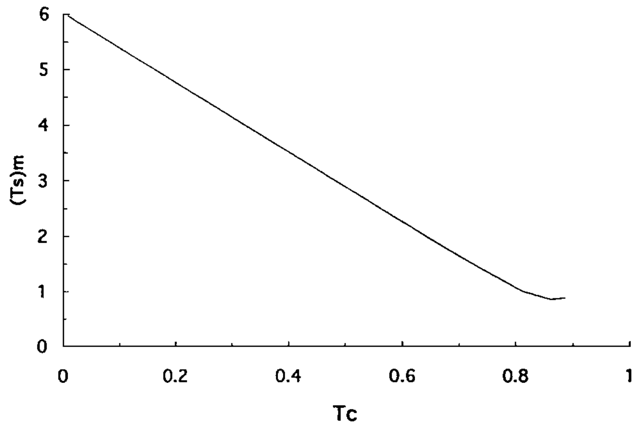

Fig. 15 Maximum shear stress in top skin $(T_s)_m$ vs shear stress in core T_c (h/h_c = 0.125).

core and the skin together must take up the shear load. The maximum shear stress developed in the skin is $(T_s)_m = 6.00$ when $T_c = 0$ and all of the shear is taken up by the skins.

The application in Sec. V was for a skin-core thickness ratio $h/h_c = 0.125$. Similar results can be obtained for other values of h/h_c, including large $h/h_c > 1$, representing two Euler beams glued together.

The normal and shear-stress distributions found here at low values of shear parameter g are similar in nature to those found by Pagano[11] for the case of a simply supported, cross-ply, composite beam under a half-sinusoidal distributed load $f = f_0 \sin \pi x/L$. For this case. which does not possess edge zones, an exact elasticity theory solution was obtained by Pagano. Because the E of the 90-deg ply of the core is small compared to the E of the 0-deg ply of the skins, the extended Timoshenko theory with boundary conditions $W = 0$, $W'' = 0$, $\alpha' = 0$ at both ends could be readily applied to give the simple deflection and rotation angle solutions of

$$W = \left(\frac{f_0 L^3}{EI}\right)\left[\frac{(3/\pi^4)(\bar{B}_1 + \bar{g})}{\bar{B}_2^2 + 3\bar{g}}\right]\sin\left(\frac{\pi x}{L}\right) \tag{96}$$

$$\alpha = \left(\frac{f_0 L^3}{EI}\right)\left[\frac{(3/\pi^3)(\bar{B}_2 - \bar{g})}{\bar{B}_2^2 + 3\bar{g}}\right]\cos\left(\frac{\pi x}{L}\right) \tag{97}$$

where $\bar{g} = g/\pi^2$. These agree well numerically with the exact solution found in Ref. 11. The corresponding stresses can similarly be developed from the preceding as before.

VI. Further Remarks

The end-loaded cantilever beam examined here can be considered as one-half of a simply supported sandwich beam of length $2L$, with a concentrated load $2P$ at its center. Thus, the analysis here also represents the familiar simply supported beam case. Extensions to other loadings are readily possible.

The present analysis is equivalent to the "antiplane core" analyses of Allen[5] and Frostig and Baruch,[6] who also arrive at sixth-order formulations, and their results are comparable to those here. However, because their formulations are in terms of deflection w and shear force Q (or w and shear stress τ) rather than the Timoshenko variables w and α, their formulations do not seem to relate simply to the familiar Timoshenko beam equations. Also the present nondimensionalization here shows that the behavior and edge zones are essentially dependent on the two nondimensional parameters h/h_c and g for these sandwich beams, and this allows for a description over a wide range of properties.

To obtain an approximate estimate of the transverse stiffness effects of a soft core E_z, one can use the superposition method proposed by Frostig and Baruch.[6] The total solution is viewed as the sum of the antiplane core solution w, α, found here (where the loads are split equally between the top and bottom skins), and an antisymmetric solution w_a, $\alpha = 0$ (where the loads act in opposite directions on the top and bottom skins). The latter solution is essentially that for a skin beam on an elastic foundation $k = 2E_c b/h_c$ and is readily obtained. For stresses at the ends of the beam, one should also consider possible additional St. Venant-type stress distributions depending on the load attachment and support details. For the present cantilever beam problem, the end load is assumed distributed as shear stress over core and skins in a predetermined fashion. In a later paper Frostig et al.[7] view the simply supported beam end as having the lower skin pinned and the upper skin free floating on the core material.

VII. Conclusions

The present paper has presented a sixth-order, extended Timoshenko theory to deal with the ambiguity of the cantilever boundary condition for sandwich beams. It represents a simple engineering step up from the familiar Timoshenko beam theory. The analysis assumes the core material of the sandwich provides shear stiffness but negligible extensional stiffness to the beam, and the two skins remain equidistant from one another (antiplane core). The equations are found to depend on essentially two nondimensional parameters, a shear parameter g, and a skin-core thickness ratio h/h_c. Trends over a wide range of shear parameter g indicate that the sandwich behavior varies smoothly from that of a conventional Euler beam at large g to two separate Euler beams of the skins at low g. In the region of the cantilever end, an edge zone develops, which transitions the beam from the conventional Timoshenko behavior to a behavior that satisfies all three boundary conditions there. Simple estimates of the extent of this edge zone are given. The edge zone also causes a redistribution of the normal and shear stresses occurring there. There may well be additional three-dimensional St. Venant-type stress distributions at the ends, depending on the details of the load attachments and the supports there.

The extended Timoshenko theory discussed here for sandwich beams is also applicable for cases of sandwich plates with soft cores (for example, see some plate formulations by Whitney[12]). The edge zones shown here to develop can also affect the vibrations of cantilever sandwich beams[3] and sandwich plates generally. The corresponding inertia terms for the vibrations of such sandwich beams are given in the Appendix. Provisions for the local buckling of the thin skins can also be considered, particularly at the cantilever root where stress concentrations develop as a result of the bending action. But here, the supporting stiffness E_z of the soft core in the vertical direction should also be considered (for example, see Refs. 5 and 9).

Appendix: Energy Formulation and Inertia Terms

The internal potential energy U for the sandwich beam shown in Fig. 1 can be expressed as

$$U = \left(\frac{1}{2}\right)\iint E(\varepsilon_x)^2 b\,dz\,dx + \left(\frac{1}{2}\right)\iint G(\gamma_{xz})^2 b\,dz\,dx \tag{A1}$$

where x varies over the length from 0 to L and z varies over the cross section from $-(h_c/2 + h)$ to $(h_c/2 + h)$ for the ε_x integral and from $-(h_c/2)$ to $(h_c/2)$ for the γ_{xz} integral. The corresponding strains from Eqs. (4) and (5), using the displacement patterns of Eqs. (1–3) become

$$(\varepsilon_x)_T = (h_c/2)\alpha' - (z - h_c/2)w'' \tag{A2}$$

$$(\gamma_{zx})_c = \alpha + w' \tag{A3}$$

$$(\varepsilon_x)_B = -(h_c/2)\alpha' - (z + h_c/2)w'' \tag{A4}$$

for the top skin, core, and bottom skin, respectively. In this Appendix the notation $(\,)'$ represents the derivative d/dx, as was used originally in Secs. II and III. Placing these strains into the internal potential energy U and carrying out the appropriate integrations gives, after much algebra,

$$U = \left(\frac{1}{2}\right)\int_0^L \left[B_3(w'')^2 - 2B_2 w''\alpha' + B_1(\alpha')^2 + GA(w' + \alpha)^2\right]dx \tag{A5}$$

If one again neglects the B_2 and B_3 terms in the preceding and replaces the B_1 with EI, one obtains the internal potential energy U of a conventional Timoshenko beam.

The work W of the applied loading f, end load P, and horizontal force F_A acting at a distance z_A above the midline, together with its oppositely directed companion on the bottom skin, is

$$W = \int_0^L f w \, dx + P w(L) + 2F_A u_T(L, z_A) \tag{A6}$$

Setting $u_T(L, z_A) = (h_c/2)\alpha(L) - (z_A - h_c/2)\, w'(L)$ from Eq. (1) in the preceding and taking the variation of the total potential energy $\Pi = U - W$ results in

$$\begin{aligned}\delta\Pi = \int_0^L &\big\{\big[B_3 w^{\mathrm{IV}} - B_2\alpha''' - GA(w'' + \alpha') - f\big]\delta w + [B_2 w''' \\ &- B_1\alpha'' + GA(w' + \alpha)]\delta\alpha\big\}\, dx + (B_3 w'' - B_2\alpha')\delta w']^L \\ &+ [-B_3 w''' + B_2\alpha'' + GA(w' + \alpha)]\delta w]^L \\ &+ (-B_2 w'' + B_1\alpha')\delta\alpha]^L - P\delta w(L) - 2F_A\left(\frac{h_c}{2}\right)\delta\alpha(L) \\ &+ 2F_A\left(z_A - \frac{h_c}{2}\right)\delta w'(L) = 0 \end{aligned} \tag{A7}$$

This gives the same differential equations, Eqs. (21) and (22), as before. The correspanding boundary conditions at $x = L$ would be

$$B_1\alpha' - B_2 w'' = h_c F_A \tag{A8}$$

$$B_3 w'' - B_2\alpha' - h_c F_A = -2z_A F_A \tag{A9}$$

$$-B_3 w''' + B_2\alpha'' + GA(w' + \alpha) = P \tag{A10}$$

from the $\delta\alpha(L)$, $\delta w'(L)$, and $\delta w(L)$ variations, respectively. Placing $h_c F_A$ from Eq. (A8) into Eq. (A9) and noting from Eq. (24) that $M_A = -2z_A F_A$ and that $S_A = -P$, one sees that these three boundary conditions are the same as Eqs. (25–27) obtained earlier from force considerations in Sec. III.

One further remark is in order regarding the energy formulation. In using energy methods to develop approximate or finite element solutions, one should be aware of the presence of edge zones that can develop near a cantilever end. The extent of these edge zones x_E can be estimated using Eqs. (66) or (66a) developed earlier.

The inertia terms for vibrations can be determined by expressing the kinetic energy T of the beam as

$$T = \left(\frac{1}{2}\right)\iint \rho_s(\dot{w}^2 + \dot{u}^2) b \, dz \, dx + \left(\frac{1}{2}\right)\iint \rho_c(\dot{w}^2 + \dot{u}^2) b \, dz \, dx \tag{A11}$$

where ρ_s and ρ_c are the densities of the skin and core respectively and the integrals are to be evaluated as before for the internal potential energy U. Noting that the vertical velocity $\dot{w}$ is considered here as constant with z and the horizontal velocity $\dot{u}$ is the time derivative of Eqs. (1–3), one can carry out the z integrations as before to obtain

$$T = \left(\frac{1}{2}\right)\int_0^L \big[m(\dot{w})^2 + (I_c + \mu_1)(\dot{\alpha})^2 - 2\mu_2\dot{\alpha}\dot{w}' + \mu_3(\dot{w}')2\big]\, dx \tag{A12}$$

where

$$m = \rho_c b h_c + 2\rho_s b h, \qquad I_c = \rho_c b h_c^3 \big/ 12, \qquad \mu_1 = (\rho_s/E)B_1$$

$$\mu_2 = (\rho_s/E)B_2, \qquad \mu_3 = (\rho_s/E)B_3 \tag{A13}$$

Placing T and U into Hamilton's principle,

$$\delta\int (T - U)\, dt = 0 \tag{A14}$$

and carrying out the variations will lead to the following equations for the vibrations of these sandwich plates:

$$B_3 w^{\mathrm{IV}} - B_2\alpha''' - GA(w'' + \alpha') = f - m\ddot{w} - \mu_2\ddot{\alpha}' + \mu_3\ddot{w}'' \tag{A15}$$

$$B_2 w''' - B_1\alpha'' + GA(w' + \alpha) = -(I_c + \mu_1)\ddot{\alpha} + \mu_2\ddot{w}' \tag{A16}$$

The vertical stiffness of the core E_z is not modeled here so that the two skins always remain equidistant from each other.

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